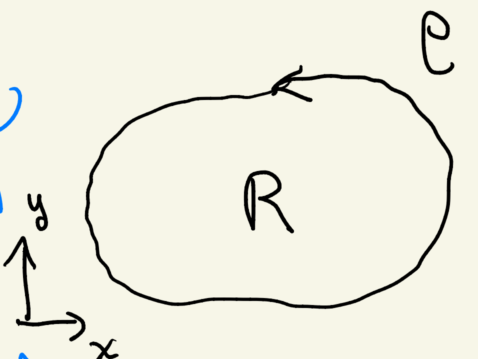
§16.4 Green's Theorem

Green's Theorem is what the Divergence Thm and Stokes Theorem both reduce to when you restrict from the real world of $(x,y,z)\in\mathbb{R}^3$ to the plane $(x,y)\in\mathbb{R}^2$

Statement of Green's Theorem.

Let $\vec{F} = \overrightarrow{(M(x,y), N(x,y))}$ be a vector field in the plane $(x,y)\in\mathbb{R}^2$, and let $\mathcal{C}$ denote a positively oriented closed curve $\mathcal{C}$. Then

$$\underbrace{\iint_R N_x - M_y \, dA}_{\text{Ch 15 double integral over } R} = \underbrace{\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\, ds}_{\text{Line Integral around the boundary of } R}$$



Comments:

- **Note** that this says that the integral of derivatives of $\vec{F}$ over a 2-dimensional region R reduces to an integral of undifferentiated components around the 1-dimensional boundary

A generalization of FTC $\quad \int_a^b f'(x)\,dx = f(b)-f(a)$

- **Note** that $N_x - M_y = \text{Curl}\,\vec{F}\cdot \underset{\sim}{k}$ if we extend $\vec{F}$ to $\mathbb{R}^3$ by making $P=0$. $\vec{F} = (M(x,y), N(x,y), 0)$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \underset{\sim}{i}\,(\cancel{P_y - N_z}) - \underset{\sim}{j}\,(\cancel{M_z - P_x}) + \underset{\sim}{k}\,(N_x - M_y)$$

$$= (N_x - M_y)\,\underset{\sim}{k}$$

Thus: $N_x - M_y = \text{Curl}\,\vec{F}\cdot\underset{\sim}{k}$ $\quad$ Put into Green's Thm

$$\underbrace{\iint_R N_x - M_y \, dA = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds}_{\text{Green's}} \Rightarrow \underbrace{\iint_R \text{Curl}\,\vec{F}\cdot\vec{n}\,dS = \oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds}_{\text{Stokes}}$$

Conclude: Green's Thm is just Stokes Thm for vector fields & curves in xy-plane

- Green's Thm is usually written with the line integral written as 1-form $M\,dx + N\,dy$

Recall:
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C \vec{F}\cdot\vec{v}\,dt \qquad \vec{v} = \frac{d\vec{r}}{dt}$$
$$= \oint_C \vec{F}\cdot d\vec{r} \qquad d\vec{r} = \vec{v}\,dt$$
$$= \oint_C \overrightarrow{(M,N)}\cdot\overrightarrow{(dx,dy)} \qquad d\vec{r} = \overrightarrow{(dx,dy)}$$
$$= \oint_C M\,dx + N\,dy$$

The standard way of writing Green's Thm is:

$$\boxed{\iint_R N_x - M_y\,dA = \oint_C M\,dx + N\,dy}$$

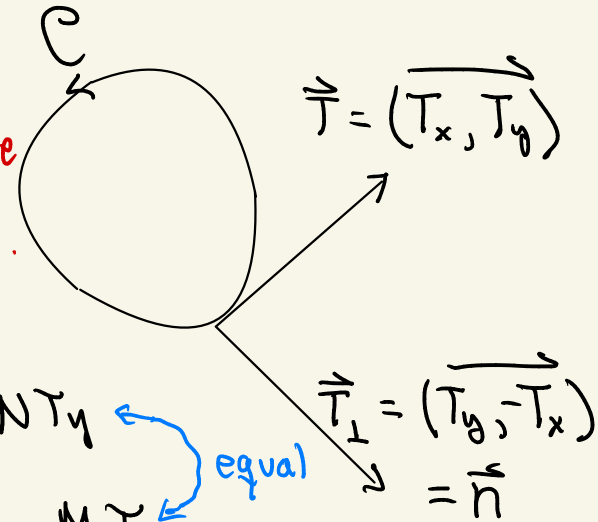
Green's Theorem

• We can also convert Green's theorem into the form of the Divergence Theorem -

Given $\vec{F} = \overrightarrow{(M,N)}$

Define $\vec{F}_\perp = \overrightarrow{(N,-M)}$

$\perp$ rotates 90° clockwise



Thus:

$$\vec{F} \cdot \vec{T} = \overrightarrow{(M,N)} \cdot \overrightarrow{(T_x, T_y)} = MT_x + NT_y$$

equal

$$\vec{F}_\perp \cdot \vec{T}_\perp = \overrightarrow{(N,-M)} \cdot \overrightarrow{(T_y, -T_x)} = NT_y + MT_x$$

$\vec{T}_\perp = \vec{n}$ = outer normal

Also: $N_x - M_y = \mathrm{Div}\,\overrightarrow{(N,-M)} = \mathrm{Div}\,\vec{F}_\perp$

So

$$\underbrace{\iint_R N_x - M_y \, dA = \oint_C M\,dx + N\,dy}_{\text{Green's Thm For } M,N \quad \vec{F} = \overrightarrow{(M,N)}} \iff \underbrace{\iint_R \mathrm{Div}\,\vec{F}_\perp \, dA = \oint_C \vec{F}_\perp \cdot \vec{n}\, ds}_{\text{Divergence Thm for } \vec{F}_\perp \quad \vec{F}_\perp = \overrightarrow{(N,-M)}}$$

Conclude: Green's Thm written in terms of $\vec{F}$ becomes the Divergence Thm when written in terms of $\vec{F}_\perp$!

Conclude: There are three equivalent forms of Green's Theorem.

(1) $$\iint_R N_x - M_y \, dA = \oint_C M\,dx + N\,dy$$ (Greens)

(2) $$\iint_R \text{Curl}\vec{F} \cdot \vec{k} \, dS = \oint_C \vec{F} \cdot \vec{T} \, ds$$ (Stokes)

(3) $$\iint_R \text{Div}\, \vec{F}_{\perp} \, dA = \oint_C \vec{F}_{\perp} \cdot \vec{n} \, ds$$ (Divergence)

$\vec{n} = \vec{T}_{\perp}$

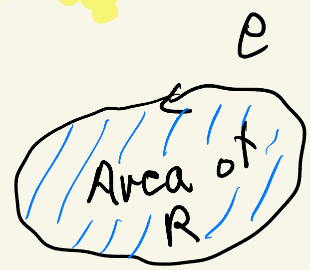
Since $\vec{F}_{\perp}$ can be any vector field, it must be true for $\vec{F}$ as well!

(3) $$\iint_R \text{Div}\, \vec{F} \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$\vec{n}$ = outer normal

Example 1 Find a vector field $\vec{F} = \langle M(x,y), N(x,y)\rangle$ such that

$$\oint_{C} \vec{F}\cdot\vec{T}\,ds = \text{Area Enclosed by } C$$



Soln: By Green's Theorem:

$$\iint_R N_x - M_y \, dA = \oint_C \vec{F}\cdot\vec{T}\,ds$$

If $N_x = \frac{1}{2}$ and $-M_y = \frac{1}{2}$, then $N_x - M_y = 1$

and $\iint_R N_x - M_y \, dA = \text{area of } R$

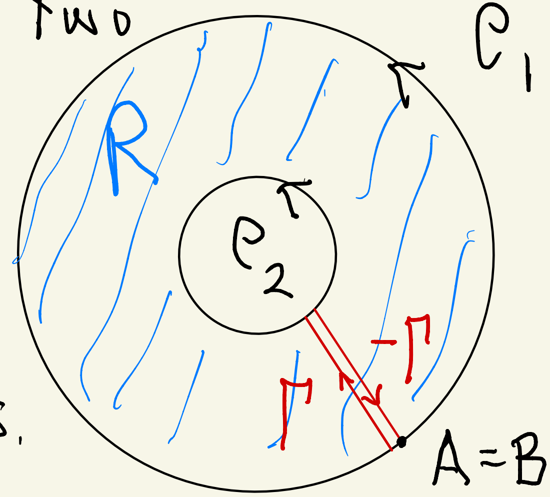
For this choose $N = \frac{1}{2}x$, $M = -\frac{1}{2}y$, $\vec{F} = \langle -\frac{1}{2}y, \frac{1}{2}x\rangle$

$$\iint_R N_x - M_y \, dA = \iint_R \tfrac{1}{2} + \tfrac{1}{2}\, dA = \iint_R dA = \text{Area of } R$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \oint_C \tfrac{1}{2}y\,dx - \tfrac{1}{2}x\,dy$$

Conclude: $\boxed{\frac{1}{2}\oint_C y\,dx - x\,dy = \text{Area Enclosed by } C}$

Example ② Consider Green's Theorem when $\vec{F}$ is defined in the annulus betw two curves C_1 & C_2. We have drawn two circles, but any two simple closed curves (scc) one inside the other works.



Show: Green's Theorem applies in the form

$$\iint_R N_x - M_y \, dA = \oint_{C_1} \vec{F}\cdot\vec{T}\, ds - \oint_{C_2} \vec{F}\cdot\vec{T}\, ds.$$

Soln. Draw in the two curves $+\Gamma$ & $-\Gamma$:

Then starting at A, $C \equiv C_1 + \Gamma - C_2 - \Gamma$ is a scc inside of which $\vec{F} = \overrightarrow{(M, N)}$ is defined.

Thus Green's Theorem applies to C:

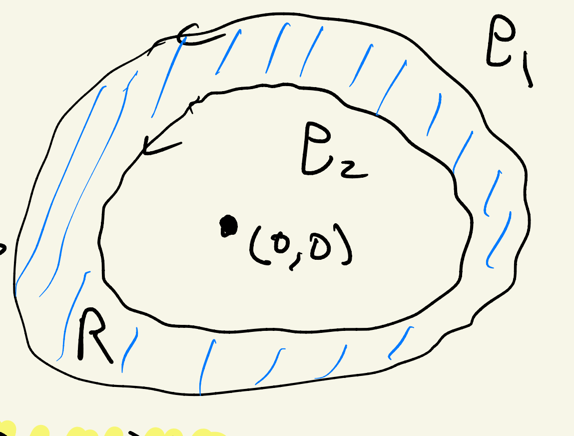
$$\iint_R N_x - M_y \, dA = \int_C \vec{F}\cdot\vec{T}\, ds = \int_{C_1 + \Gamma - C_2 - \Gamma} \vec{F}\cdot\vec{T}\, ds$$

$$= \int_{C_1} + \cancel{\int_{\Gamma}} - \int_{C} - \cancel{\int_{\Gamma}} = \oint_{C_1} \vec{F}\cdot\vec{T}\, ds - \oint_{C_2} \vec{F}\cdot\vec{T}\, ds \quad \checkmark$$

Example 3: Use **Example 2** to show that if $\text{Curl}\vec{F} = 0$ in $D = \{(x,y) : (x,y) \neq 0\} = \mathbb{R}^2 \setminus \{0,0\}$, then $\oint_{C_1} \vec{F}\cdot\vec{T}\,ds = \oint_{C_2} \vec{F}\cdot\vec{T}\,ds$ for any two positively oriented curves C_1, C_2 which go around $(0,0)$ exactly once.

Solution: Since D is not simply connected, we cannot conclude from $\text{Curl}\vec{F} = 0$ that $\vec{F}$ is conservative, $\vec{F} = \nabla f$, or that the line integral $\oint_C \vec{F}\cdot\vec{T}\,ds$ around closed curves $= 0$.

Alternatively, apply Green's Theorem in Annulus between C_1 & C_2:



$$0 = \iint_R \underbrace{\text{Curl}\vec{F}\cdot\vec{n}}_{N_x - M_y \to 0}\, dA = \oint_{C_1} \vec{F}\cdot\vec{T}\,ds - \oint_{C_2} \vec{F}\cdot\vec{T}\,ds$$

so $\oint_{C_1} \vec{F}\cdot\vec{T}\,ds = \oint_{C_2} \vec{F}\cdot\vec{T}\,ds$ ✓